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Lévy-Driven Langevin Systems: Targeted Stochasticity

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Langevin dynamics driven by random Wiener noise ("white noise"), and the resulting Fokker–Planck equation and Boltzmann equilibria are fundamental to the understanding of transport and relaxation. However, there is experimental and theoretical evidence that the use of the Gaussian Wiener noise as an underlying source of randomness in continuous time systems may not always be appropriate or justified. Rather, models incorporating general Lévy noises, should be adopted.In this work we study Langevin systems driven by general Lévy, rather than Wiener, noises. Various issues are addressed, including: (i) the evolution of the probability density function of the system's state; (ii) the system's steady state behavior; and, (iii) the attainability of equilibria of the Boltzmann type. Moreover, the issue of *reverse engineering* is introduced and investigated. Namely: how to design a Langevin system, subject to a given Lévy noise, that would yield a pre-specified "target" steady state behavior. Results are complemented with a multitude of examples of Lévy driven Langevin systems.

KEY WORDS: Langevin dynamics; Lévy noise; Fokker–Planck equation; Boltzmann equilibria; reverse engineering.

1. INTRODUCTION

Langevin systems driven by a Gaussian (white) noise have been studied extensively in the literature. These systems are governed by dynamics of the type

$$X(dt) = -U'(X(t))\,dt + \sigma W(dt), \tag{1}$$

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where: (i) U' is the derivative of an external potential U; (ii) σ is a positive constant representing the noise amplitude; and, (iii) $W = (W(t))_{t \geqslant 0}$—a Wiener process whose derivative, $\dot{W}$, is the "white noise"—is the underlying source of randomness driving the system. One of the main tools used to investigate the Langevin dynamics (1) is the Fokker–Planck equation which governs the evolution of the probability density function (pdf) of the system's state.[1–4]

The Wiener process however, is a special case within the family of stochastic processes qualifying as natural models for random noise sources. This is the family of Lévy processes, introduced and pioneered by Paul Lévy[5–7] (see also refs. 8–17). The increments of a general Lévy process satisfy two key properties: (i) they are stationary (shift invariant); and, (ii) non-overlapping increments are independent. These features manifest the intuitive meaning of "noise".

Wiener Noise vs Lévy Noise

One of the main features distinguishing the Gaussian Wiener process from all other non-Gaussian Lévy processes is the continuity of its sample paths: the trajectories of the Wiener process are continuous (though, nowhere differentiable), whereas the trajectories of non-Gaussian Lévy processes are purely discontinuous. In other words, in the Gaussian case propagation is conducted continuously via diffusion, while in the non-Gaussian cases propagation is conducted discontinuously and discretely via jumps.

Another major distinction between the Wiener process and non-Gaussian Lévy processes is intimately related to the issue of scale invariance. As described above, a key property of Lévy processes is the shift invariance of their increments. However, if we wish the noise model to be invariant not only under shifts but also under changes of scale—that is, if we wish the noise to be of a fractal nature—then an additional requirement of scale invariance is to be posed. It turns out that amongst the sub-class of scale invariant Lévy processes, the Wiener process is the only one with finite variance—all other scale invariant processes have infinite variance.[8–11]

The special and unique features of the Wiener process—continuity of sample paths, scale invariance, convergence of moments—allowed for the development of powerful and tractable analytical tools, such as the Fokker–Planck equation, the Feynman–Kac formula, and the celebrated Ito calculus[18–21] (see also refs. 22–25). This, in turn, was one of the major reasons turning the Wiener process to serve as a natural, if not almost unique, choice for modeling random noise in continuous time systems.

In recent years Lévy process have drawn much attention and research.[26–40] On the one hand, numerous examples and evidence of non-Gaussian noises have been discovered and documented in many "real-world" complex systems. In fact, statistics of the Lévy type turned out to be ubiquitous a phenomena empirically observed in various areas including: physics (anomalous diffusion, turbulent flows, nonlinear Hamiltonian dynamics[29, 34]), biology (heartbeats,[41] firing of neural networks[42]) seismology (recordings of seismic activity[43]), electrical engineering (signal processing[44–46]), and economics (financial time series[47–49]) (for further examples see refs. 31, 36, and 39 and references therein). On the other hand, the ruling paradigm of modeling noise in continuous time stochastic systems as Gaussian, began to give way to the examination and incorporation of models driven by non-Gaussian noises.

Moreover, recently introduced kinetic equations with fractional space and time derivatives have attracted attention as possible a tool for the description of anomalous diffusion and relaxation phenomena.[35, 38, 40] These fractional Fokker–Planck equations turn out to be the analytical analog of the Fokker–Planck equation in cases where the source of randomness is a non-Gaussian scale invariant continuous time noise. That is, when we change the system's underlying noise source from the Gaussian Wiener process to non-Gaussian scale invariant Lévy processes, the Fokker–Planck partial differential equation (governing the evolution of the pdf of the system's state) needs to be modified into a fractional equation—a partial differential equation containing non-integer derivatives. The degree of the fractional derivative, as may be suspected, is tightly related to the "fractal dimension" of the underlying scale invariant Lévy process.[8–11]

Lévy Driven Langevin Systems

Lévy driven Langevin systems are Langevin equations of the type (1), where the driving Wiener process is replaced by a non-Gaussian Lévy process. Namely, let $X = (X(t))_{t \geqslant 0}$ be a stochastic process governed by the following dynamics

$$X(dt) = \underbrace{-f(X(t))\,dt}_{\text{Drift}} + \underbrace{L(dt)}_{\text{Driver}}, \tag{2}$$

where: (i) $-f = -U'$ is the system's drift function, stemming from an external potential U; and, (ii) $L = (L(t))_{t \geqslant 0}$ is a non-Gaussian Lévy process serving as the underlying source of randomness driving the system. Dynamics of the type (2) were explored in refs. 27, 30, 37, 38, and 40.

Note that the "drift" and the "driver" are of completely orthogonal nature: the former being continuous, deterministic, and predictable, while the latter is discontinuous, random, and unpredictable. Note also that the Langevin dynamics (2) could be regarded as a non-linear version of the Orenstein–Uhlenbeck dynamics,[1] in which the drift function f is linear (the potential is quadratic).

Two types of systems are of special interest—*symmetric* and *subordinate*:

In *symmetric systems*, the stochastic process X is symmetric with respect to the origin. That is, the potential U and the Lévy driver L are symmetric (or, equivalently, the drift function f is anti-symmetric and the Lévy driver L is symmetric).

In *subordinate systems*, the stochastic process X is non-negative valued, i.e., the Langevin dynamics (2) take place on the non-negative half line $[0, \infty)$. In this case the drift function f is positive, and the Lévy driver L is a Lévy subordinate, i.e., a Lévy process with non-negative increments (or, in other words, a pure-jump Lévy process with positive jumps).

Subordinate Systems

Subordinate systems can be visualized as a model of particle motion on the non-negative half line, where the particle is subjected to two "rivaling" forces acting simultaneously: (i) random "kicks"—appearing at random times, and of random magnitude—which are "kicking" the particle towards ∞; and, (ii) a state-dependent deterministic drift pushing the particle towards the origin 0. The system's stationary pdf emerges as a "compromise" between these two opposing forces.

Alternatively, subordinate systems can be visualized as a reservoir or dam model: X representing the level of water in the reservoir, L representing the random rainfall, and f representing the deterministic pumping-out intensity (dependent on the level of the water in the reservoir).

In this work we study the Langevin equation (2), driven by general non-Gaussian Lévy noises, and focus on the following issues:

1. **Evolution:** *What is the Fokker–Planck equation governing the evolution of the pdf of the system's state?*

2. **Steady state:** *In steady state, what is the connection between the system's drift function f, driving noise, and stationary pdf?*

3. **Reverse engineering:** *Given a "target" pdf p, can we "tailor design" a drift function f so that the system's stationary pdf would equal the desired "target" pdf p?*

4. **Boltzmann equilibria:** It is well know that in Wiener-driven Langevin dynamics, i.e., in the Gaussian case (1), the system admits a Boltzmann equilibrium. Namely, the system's stationary pdf equals

$$c \exp\left\{ -\frac{2}{\sigma^2} U(x) \right\}, \tag{3}$$

where c is a normalizing constant, σ is the noise amplitude, and U is the external potential. Hence, the following question arises naturally: *Are Boltzmann-type equilibria still attainable when the Lévy driver is non-Gaussian?*

The paper is organized as follows; In Section 2 we describe, in detail, the properties and structure of the driving (pure-jump, non-Gaussian) Lévy process. In Section 3 the Langevin dynamics (2) are analyzed, and various issues—the Fokker–Planck evolution equation, steady state, systems with polynomial drift functions, and existence of Boltzmann-type equilibria—are addressed. In Section 4 the issue of reverse engineering is investigated. We conclude, in Section 5, with a multitude of examples.

1.1. Notations

The following notations will be used throughout the paper:

Fourier transform: $\hat{\varphi}$ will denote the Fourier transform of a function φ (defined on the real line);

$$\hat{\varphi}(\omega) = \int_{-\infty}^{\infty} \exp\{i\omega x\}\, \varphi(x)\, dx.$$

Laplace transform: $\tilde{\varphi}$ will denote the Laplace transform of a non-negative valued function φ (defined on the non-negative half line);

$$\tilde{\varphi}(\omega) = \int_{0}^{\infty} \exp\{-\omega x\}\, \varphi(x)\, dx; \qquad \omega \geqslant 0.$$

Convolution: $\varphi_1 * \varphi_2$ will denotes the convolution of the functions φ_1, φ_2 (defined on the real line);

$$(\varphi_1 * \varphi_2)(x) = \int_{-\infty}^{\infty} \varphi_1(x-y)\, \varphi_2(y)\, dy.$$

Also, the symbol $\sim$ will denote equality in distribution, and $\approx$ will denote asymptotic equivalence:

$$\varphi_1 \approx \varphi_2 \ (x \to l) \ \Leftrightarrow \ \lim_{x \to l} \frac{\varphi_1(x)}{\varphi_2(x)} = 1,$$

for the functions φ_1, φ_2 (defined on the real line).

Finally, $\mathbf{P}(E)$ will denote the probability of an event E, and $\mathbf{E}[R]$ will denote the expectation of a random variable R.

2. THE LÉVY DRIVER

2.1. The Fourier and Laplace Characteristics

Mathematically, the family of Lévy processes consists of all stochastic processes with independent and stationary increments, which are continuous in probability. A Lévy process $L = (L(t))_{t \geqslant 0}$ is characterized by it's spectral (Fourier) representation

$$\mathbf{E}[\exp\{i\omega L(t)\}] = \exp\{-\Psi(\omega) \cdot t\}. \tag{4}$$

The function Ψ is called the Fourier characteristic of the Lévy process L (in the literature, Ψ is also referred to as the spectral characteristic, or symbol, of the process L).

When L is a Lévy subordinate, i.e., a Lévy process with non-negative increments, then it can be characterized, alternatively, by the Laplace (rather that Fourier) representation

$$\mathbf{E}[\exp\{-\omega L(t)\}] = \exp\{-\Phi(\omega) \cdot t\}; \qquad \omega \geqslant 0. \tag{5}$$

The function Φ is referred to as the Laplace characteristic of the Lévy subordinate L.

2.2. Self Similar Lévy Processes

Amongst the Lévy family, the class of self similar processes is of special importance. This class consists of all scale invariant (fractal) processes. A Lévy process $L = (L(t))_{t \geqslant 0}$ is said to be self similar with exponent α (or, in short, α-self similar) if the following condition holds: $\forall$ positive constant $k > 0$ the k-scaled process $L^{(k)} = (\frac{1}{k} L(k^{\alpha} t))_{t \geqslant 0}$ is equal, in distribution, to the original process L. It turns out that:

(i) if L is symmetric then

$$\Psi(\omega) = a\, |\omega|^{\alpha}, \tag{6}$$

where $a > 0$ is the (noise) amplitude. The admissible values of the exponent are $0 < \alpha \leqslant 2$ ($\alpha = 1$ and $\alpha = 2$ yielding, respectively, the Cauchy and Gaussian distributions).

(ii) if L is a subordinate then

$$\Phi(\omega) = a\omega^{\alpha}, \tag{7}$$

where $a > 0$ is the (noise) amplitude. The admissible values of the exponent are $0 < \alpha < 1$ ($\alpha = \frac{1}{2}$ yielding the Lévy-$\frac{1}{2}$ distribution).

2.3. Pure-Jump Lévy Processes and Poisson Superpositions

A pure-jump Lévy process can be viewed, formally, as a superposition of a continuum of independent Poisson processes. Indeed:

If L is a Poisson process with jumps of size x_0 and rate λ_0, then it is a Lévy process and its Fourier characteristic is $\Psi(\omega) = (1 - \exp\{i\omega x_0\})\, \lambda_0$. If L is a superposition of N independent Poisson processes—process n, $n = 1, 2, \ldots, N$, having jumps of size x_n and rate λ_n—then, again, it is a Lévy process and its Fourier characteristic is given by

$$\Psi(\omega) = \sum_{n=1}^{N} (1 - \exp\{i\omega x_n\})\, \lambda_n. \tag{8}$$

Hence, passing from (8) to the continuum limit, where jumps of size x occur at rate $\lambda(dx)$, we arrive at

$$\Psi(\omega) = \int_{-\infty}^{\infty} (1 - \exp\{i\omega x\})\, \lambda(dx), \tag{9}$$

pending on the convergence of the right hand side of (9).

The rate $\lambda(\cdot)$—henceforth referred to as the process' *jump measure*—is a measure on the real line. The jump measure might have an infinite total mass—not due to divergence at $|x| \to \infty$ but rather, due to a possible divergence at $|x| \to 0$ (intuitively, large jumps can occur only rarely, but tiny jumps may occur very frequently). The rigorous formalism of (9) is provided by the celebrated Lévy–Khinchin formula.[8–11]

2.4. Compound Poisson Process

If the jump measure is finite, i.e., if $\int_{-\infty}^{\infty} \lambda(dx) < \infty$, then L is a compound Poisson process with rate $r = \int_{-\infty}^{\infty} \lambda(dx)$ and jump-size J distributed according to the probability measure $\frac{1}{r}\, \lambda(dx)$. Namely:

$$L(t) = \sum_{n=1}^{N(rt)} J_n, \tag{10}$$

where: (i) $N=(N(t))_{t\geqslant 0}$ is a standard Poisson process (i.e., with rate 1); (ii) $\{J_n\}_{n=1}^{\infty}$ is the sequence of independent and identically distributed jumps (with probability measure $\frac{1}{r}\lambda(dx)$); and (iii) The Poisson process and the sequence of jumps are mutually independent.

The converse is also true: if (i)–(iii) above hold, then the process $L=(L(t))_{t\geqslant 0}$ given by (10) is a Lévy process with jump measure $\lambda(dx)=r\mathbf{P}(J\in dx)$. Moreover:

$$\Psi(\omega)=r(1-\mathbf{E}[\exp\{i\omega J\}]), \tag{11}$$

and, in the case of non-negative jumps (implying that L is a subordinate):

$$\Phi(\omega)=r(1-\mathbf{E}[\exp\{-\omega J\}]). \tag{12}$$

2.5. Examples

We mention a few examples:

1. If $\lambda(dx)=c_\alpha\cdot|x|^{-(1+\alpha)}\,dx$, where $0<\alpha<2$ and c_α is an appropriately chosen normalizing constant, then L is a symmetric α-self similar Lévy process with unit amplitude ($\Psi(\omega)=|\omega|^\alpha$).

2. If $\lambda(dx)=c_\alpha\cdot x^{-(1+\alpha)}\,dx$, $x>0$, where $0<\alpha<1$ and c_α is an appropriately chosen normalizing constant, then L is an α-self similar Lévy subordinate with unit amplitude ($\Phi(\omega)=\omega^\alpha$).

3. If $\lambda(dx)=\frac{\exp\{-\mu|x|\}}{|x|}\,dx$, where $\mu>0$, then $\Psi(\omega)=\ln(1+(\frac{\omega}{\mu})^2)$.

4. If $\lambda(dx)=\frac{\exp\{-\mu x\}}{x}\,dx$, $x>0$, where $\mu>0$, then L is a Gamma subordinate, i.e., its increments obey the Gamma distribution. This subordinate is characterized by $\Phi(\omega)=\ln(1+\frac{\omega}{\mu})$.

5. If $\lambda(dx)=\frac{r}{\pi}(1+x^2)^{-1}\,dx$, where $r>0$, then L is a compound Poisson process with rate r and Cauchy jumps.

6. If $\lambda(dx)=r\mu\exp\{-\mu x\}\,dx$, $x>0$, where $r,\mu>0$, then L is a compound Poisson subordinate with rate r and exponential jumps.

2.6. Finite Jump Rate vs Finite Variance

From the examples it should be evident that the issue of a finite jump measure and the issue of a finite variance of the Lévy process are separate and unrelated. The finiteness of the jump measure depends on its integrability at $|x|\to 0$, whereas the converges of variance depends on the integrability of $x^2\lambda(dx)$ at $|x|\to\infty$. Hence a Lévy process could be a compound Poisson process with an infinite variance, e.g., when the jumps are

Cauchy (example 5 above), and it can be a process with infinite jump measure but with a finite variance, e.g., the Gamma process (example 4 above).

2.7. The Tail Function

Given a pure-jump Lévy process L with jump measure $\lambda(\cdot)$, we define the process' tail function Λ to be:

$$\Lambda(x) = \begin{cases} \int_x^\infty \lambda(du) & x > 0 \\ -\int_{-\infty}^x \lambda(du) & x < 0. \end{cases} \tag{13}$$

If L is a subordinate process then $\Lambda(x) = \int_x^\infty \lambda(du)$, $x > 0$.

The tail function and the Fourier and Laplace characteristics of L are related to each other via

$$\Psi(\omega) = -i\omega\hat{\Lambda}(\omega), \tag{14}$$

and, in the subordinate case, via

$$\Phi(\omega) = \omega\tilde{\Lambda}(\omega). \tag{15}$$

The proofs of (14) and (15) are given in the appendix.

3. ANALYSIS OF THE LANGEVIN SYSTEM

3.1. The Infinitesimal Generator

The infinitesimal generator, **A**, of the Langevin system (2) is given by

$$(\mathbf{A}\varphi)(x) = \underbrace{-f(x)\,\varphi'(x)}_{(\mathbf{D}\varphi)(x)} + \underbrace{\int_{-\infty}^{\infty} \{\varphi(x+y) - \varphi(x)\}\,\lambda(dy)}_{(\mathbf{J}\varphi)(x)}, \tag{16}$$

$\forall \varphi \in \mathscr{D}(\mathbf{A})$, where $\mathscr{D}(\mathbf{A})$ denotes the domain of the operator **A**.

The differential operator **D** stems from the continuous and deterministic drift of the dynamics (2), whereas the integral operator **J** arises from the discontinuous and random jumps of the Lévy driver L.

3.2. The Fokker–Planck Equation

Let $P(t,\cdot)$, $t \geqslant 0$, denote the pdf of $X(t)$—the system's state at time t (that is; $\mathbf{P}(X(t) \in dx) = P(t, x)\,dx$). The evolution of $(P(t,\cdot))_{t\geqslant 0}$ is given by

$$\frac{\partial P}{\partial t} = \mathbf{A}^* P, \tag{17}$$

where $\mathbf{A}^*$ is the adjoint operator of the infinitesimal generator $\mathbf{A}$.

Theorem 1. The adjoint operator, $\mathbf{A}^*$, of the infinitesimal generator $\mathbf{A}$ is given by

$$\mathbf{A}^* p = \frac{\partial}{\partial x}(f \cdot p - \Lambda * p), \tag{18}$$

$\forall$ pdf p.

(See the appendix for the proof of theorem 1)

Combining (17) and (18) together yields the Fokker–Planck equation for the dynamics (2):

$$\frac{\partial P}{\partial t} = \frac{\partial}{\partial x}(f \cdot P - \Lambda * P). \tag{19}$$

It is illuminating to compare (19) with the Fokker–Planck equation of the Wiener-driven Langevin dynamics (1):[1, 3]

$$\frac{\partial P}{\partial t} = \frac{\partial}{\partial x}\left(f \cdot P - \frac{\sigma^2}{2}\frac{\partial P}{\partial x}\right). \tag{20}$$

The second order differential operator $\frac{\sigma^2}{2}\frac{\partial^2}{\partial x^2}(\cdot)$ in the Fokker–Planck equation, stemming from the Gaussian driver σW, is replaced by the integro-differential operator $\frac{\partial}{\partial x}(\Lambda * (\cdot))$ arising from the pure-jump Lévy driver L. Alternatively, the differential "flux operator" $(f \cdot - \frac{\sigma^2}{2}\frac{\partial}{\partial x})(\cdot)$ in the Gaussian case is replaced by the integral "flux operator" $(f \cdot - \Lambda *)(\cdot)$ in the non-Gaussian Lévy case.

It is also interesting to note that when the driver is an α-self similar Lévy process ($0 < \alpha < 2$) then the integro-differential operator $\frac{\partial}{\partial x}(\Lambda * (\cdot))$ is equivalent to the fractional derivative $\frac{\partial^\alpha}{\partial x^\alpha}(\cdot)$ used in refs. 38 and 40.

3.3. Stationarity

A pdf p is said to be a stationary pdf for the stochastic system $(X(t))_{t\geqslant 0}$ if and only if $X(0) \sim p \Rightarrow X(t) \sim p\ \forall t > 0$.

Theorem 2. A probability density function p is a stationary density function for the Langevin system (2) if and only if

$$f \cdot p = \Lambda * p. \tag{21}$$

(See the appendix for the proof of theorem 2)

Since $(\widehat{\Lambda * p})(\omega) = \hat{\Lambda}(\omega) \cdot \hat{p}(\omega) = -\frac{\Psi(\omega)}{i\omega} \cdot \hat{p}(\omega)$, transforming to Fourier domain implies that (21) is equivalent to

$$\frac{\widehat{fp}(\omega)}{\hat{p}(\omega)} + \frac{\Psi(\omega)}{i\omega} = 0, \tag{22}$$

and, in the symmetric case, to

$$\frac{\int_0^\infty (fp)(x)\sin(\omega x)\,dx}{\int_0^\infty p(x)\cos(\omega x)\,dx} = \frac{\Psi(\omega)}{\omega}. \tag{23}$$

In the subordinate case $(\widetilde{\Lambda * p})(\omega) = \tilde{\Lambda}(\omega) \cdot \tilde{p}(\omega) = \frac{\Phi(\omega)}{\omega} \cdot \tilde{p}(\omega)$. Hence transforming to Laplace domain implies that (21) is equivalent to

$$\frac{\widetilde{fp}(\omega)}{\tilde{p}(\omega)} = \frac{\Phi(\omega)}{\omega}. \tag{24}$$

It should be noted that Eq. (22) enables us to estimate the Fourier characteristic Ψ of the system's noise. Noise estimation is of interest in cases where the system's noise is "internal" and unobservable, but where the system's drift function f is known. Indeed, monitoring the system's state $(X(t))_{t\geqslant 0}$ one can obtain an empirical stationary pdf, and using (22) one can hence estimate the Fourier characteristic Ψ of the unobservable Lévy driver L.

3.4. Polynomial Drift Functions

When the drift function f is a polynomial, the Fourier and Laplace transforms of the stationary pdf p satisfy linear ordinary differential equations (ODEs), as the following propositions assert:

Symmetric Case

Proposition 3. In a symmetric system the following statements are equivalent:

(i) The drift function is a polynomial

$$f(x) = \sum_{n=1}^{N} c_n \cdot x^{2n-1}. \tag{25}$$

(ii) The Fourier transform of the stationary pdf satisfies the linear ODE

$$\sum_{n=1}^{N} c_n(-1)^n \cdot \frac{\partial^{2n-1}}{\partial \omega^{2n-1}} \hat{p}(\omega) = \frac{\Psi(\omega)}{\omega} \cdot \hat{p}(\omega). \tag{26}$$

Proof. In the symmetric case $\hat{p}(\omega) = 2\int_0^\infty p(x)\cos(\omega x)\,dx$ and hence, $\forall$ integer n;

$$(-1)^n \cdot \frac{\partial^{2n-1}}{\partial \omega^{2n-1}} \hat{p}(\omega) = 2\int_0^\infty x^{2n-1} p(x)\sin(\omega x)\,dx, \tag{27}$$

which, in turn, implies that

$$\sum_{n=1}^{N} c_n(-1)^n \cdot \frac{\partial^{2n-1}}{\partial \omega^{2n-1}} \hat{p}(\omega) = 2\int_0^\infty \left(\sum_{n=1}^{N} c_n \cdot x^{2n-1}\right) p(x)\sin(\omega x)\,dx. \tag{28}$$

Combining (28) and (23) together implies that (25) and (26) are equivalent. ∎

Subordinate Case

Proposition 4. In a subordinate system the following statements are equivalent:

(i) The drift function is a polynomial

$$f(x) = \sum_{n=1}^{N} c_n \cdot x^n. \tag{29}$$

(ii) The Laplace transform of the stationary pdf satisfies the linear ODE

$$\sum_{n=1}^{N} c_n(-1)^n \cdot \frac{\partial^n}{\partial \omega^n} \tilde{p}(\omega) = \frac{\Phi(\omega)}{\omega} \cdot \tilde{p}(\omega). \tag{30}$$

Proof. In the subordinate case $\tilde{p}(\omega) = \int_0^\infty p(x) \exp\{-\omega x\}\, dx$ and hence, $\forall$ integer n;

$$(-1)^n \cdot \frac{\partial^n}{\partial \omega^n} \tilde{p}(\omega) = \int_0^\infty x^n p(x) \exp\{-\omega x\}\, dx, \tag{31}$$

which, in turn, implies that

$$\sum_{n=1}^{N} c_n (-1)^n \cdot \frac{\partial^n}{\partial \omega^n} \tilde{p}(\omega) = \int_0^\infty \left(\sum_{n=1}^{N} c_n \cdot x^n \right) p(x) \exp\{-\omega x\}\, dx. \tag{32}$$

Combining (32) and (24) implies that (29) and (30) are equivalent. ∎

Propositions 3 and 4 transform the, rather difficult, convolution equation (21) to the, simpler and more tractable, linear ODEs (26) and (30).

When the drift function is linear, i.e., when the dynamics are of the Orenstein–Uhlenbeck type,[(1)] the resulting equations (26) and (30) are first order linear ODEs, and their explicit solutions are easily computed. Indeed, if $f(x) = ax$ $(a > 0)$ then

$$\hat{p}(\omega) = \exp\left\{ -\frac{1}{a} \int_0^{|\omega|} \frac{\Psi(\mathrm{sign}(\omega) \cdot u)}{u}\, du \right\}, \tag{33}$$

in the symmetric case, and

$$\tilde{p}(\omega) = \exp\left\{ -\frac{1}{a} \int_0^{\omega} \frac{\Phi(u)}{u}\, du \right\}, \tag{34}$$

in the subordinate case.

It should be noted that an alternative derivation of (33) and (34) can be obtained via a direct analysis of the linear Orenstein–Uhlenbeck equation $X(dt) = -aX(t)\, dt + L(dt)$.

3.5. Boltzmann-Type Equilibria

As for the existence of Boltzmann-type equilibria—the following proposition *excludes* their possibility in Lévy driven Langevin systems:

Proposition 5. Boltzmann-type equilibria in the Langevin system (2) are **non**-attainable when the Lévy driver is purely non-Gaussian.

Proof. Assume that the stationary pdf p of the Langevin system (2) is of a Boltzmann-type, i.e., that it is of the form

$$p(x) = c\exp\left\{-\frac{1}{k}U(x)\right\}, \tag{35}$$

where c is a normalizing constant, k is some positive constant, and U is the potential $(-f = -U')$.

Differentiating (35), $p' = p\cdot(-\frac{1}{k}U') = -\frac{1}{k}\cdot(f\cdot p)$, gives $-k\cdot p' = f\cdot p$. Hence, $\widehat{fp}(\omega) = -k\cdot\widehat{p'}(\omega) = k\cdot i\omega\hat{p}(\omega)$, which, in turn, yields

$$\frac{\widehat{fp}(\omega)}{\hat{p}(\omega)} = k\cdot i\omega. \tag{36}$$

Substituting (36) into (22) we obtain

$$\Psi(\omega) = k\cdot\omega^2. \tag{37}$$

However, (37) implies that $L = \sqrt{2k}\cdot W$ where $W = (W(t))_{t\geqslant 0}$ is a Wiener process—in strict contradiction to the non-Gaussianity of the Lévy driver! ∎

4. REVERSE ENGINEERING

4.1. Reverse Engineering and System Reconstruction

The drift function f is easily extracted out of Eq. (21) giving

$$f = \frac{\Lambda * p}{p}. \tag{38}$$

In the symmetric case (38) becomes

$$f(x) = \int_0^\infty \Lambda(u)\frac{p(x-u) - p(x+u)}{p(x)}\,du, \tag{39}$$

and, in the subordinate case,

$$f(x) = \int_0^x \Lambda(u)\frac{p(x-u)}{p(x)}\,du. \tag{40}$$

Formula (38) is of major importance, since it enables us to either *reverse engineer* or *reconstruct* the system:

Reverse engineering: Given a "target" pdf p, formula (38) hands us the drift function that would yield a stationary pdf equaling p. In other words, (38) tells us how to "tailor-design" the system (—the drift function f) in order to obtain a pre-specified 'target' stationary behavior (—the pdf p).

Reconstruction: Having a "black-box" system—a system whose evolution we monitor, but do not know its "internal mechanism," i.e., its drift function f—formula (38) enables us to perform system reconstruction. That is, (38) tells us how to reconstruct the unknown drift function f from the observed system states. Indeed, monitoring the system's state $(X(t))_{t\geqslant 0}$ one can obtain an empirical stationary pdf, and using (38) one can therefore estimate the system's drift function f.

Henceforth, we will focus on the issue of reverse engineering.

Noise Confinement

The reverse engineering formula is also a "recipe" for the confinement of noise. When designing a system, it is often desired to design it so that its output randomness, i.e., the randomness of its stationary pdf, is smaller than the randomness of its driving noise. By "randomness" of a pdf we mean the heaviness of its tails: the heavier the tails—the more "wild" and random the distribution, the lighter the tails—the more "tamed" is the distribution. In other words, noise confinement makes the tails steeper. This turns out to be a straightforward task using the reverse engineering formula: simply set the "target" pdf to have the required light tails—the obtained drift function will yield the desired noise confinement.

4.2. Fourier and Laplace Representations

The hard part of the reverse engineering procedure lies in the computation of the convolution $g = \Lambda * p$. The Fourier transform of g, however, is easily calculated:

$$\hat{g}(\omega) = -\frac{\Psi(\omega)}{i\omega} \cdot \hat{p}(\omega). \tag{41}$$

In the symmetric case (41) reduces to:

$$\int_0^\infty g(x)\sin(\omega x)\,dx = \frac{\Psi(\omega)}{2\omega} \cdot \hat{p}(\omega), \tag{42}$$

or, alternatively (using the Fourier inversion formula);

$$g(x) = \frac{1}{\pi}\int_0^\infty \frac{\Psi(\omega)}{\omega}\,\hat{p}(\omega)\sin(\omega x)\,d\omega, \tag{43}$$

and hence, also

$$f(x) = \frac{\int_0^\infty \frac{\Psi(\omega)}{\omega}\, \hat{p}(\omega) \sin(\omega x)\, d\omega}{\int_0^\infty \hat{p}(\omega) \cos(\omega x)\, d\omega}. \tag{44}$$

In the subordinate case, the Laplace transform of g is given by

$$\tilde{g}(\omega) = \frac{\Phi(\omega)}{\omega} \cdot \tilde{p}(\omega). \tag{45}$$

4.3. Self Similar Noise

When the Lévy driver is self similar, the use of Tauberian theorems[50] leads to explicit asymptotic formulae for the reverse-engineered drift functions:

Symmetric Case

Proposition 6. Assume a symmetric system driven by an α-self similar noise with amplitude a, i.e., $\Psi(\omega) = a\,|\omega|^\alpha$, where $a > 0$ and $0 < \alpha < 2$, $\alpha \neq 1$. Then, the asymptotics of the reverse-engineered drift function yielding stationary pdf p are given by

$$f(x) \approx c_\alpha \cdot \frac{a}{|x|^\alpha\, p(x)} \qquad (|x| \to \infty), \tag{46}$$

where $c_\alpha = (1-\alpha)/2\Gamma(2-\alpha)\cos(\frac{\pi}{2}\alpha)$.

Proof. By (42) we have

$$\int_0^\infty g(x) \sin(\omega x)\, dx = \frac{a}{2}\,\omega^{\alpha-1} \cdot \hat{p}(\omega) \approx \frac{a}{2}\,\omega^{\alpha-1} \qquad (\omega \to +0). \tag{47}$$

Hence, if $0 < \alpha < 1$, then, by Pitman's Tauberian theorem (see ref. 50, Thm. 4.10.3)

$$g(x) \approx c_1 \frac{a}{x^\alpha} \qquad (x \to \infty), \tag{48}$$

where $c_1 = 1/2\Gamma(1-\alpha)\cos(\frac{\pi}{2}\alpha)$.

If $1 < \alpha < 2$ then differentiating (47) gives

$$\int_0^\infty x g(x) \cos(\omega x)\, dx \approx \frac{a(\alpha-1)}{2}\,\omega^{(\alpha-1)-1} \qquad (x \to \infty), \tag{49}$$

and again, using Pitman's Tauberian theorem,

$$g(x) \approx c_2 \frac{a}{x^{\alpha}} \qquad (x \to \infty), \tag{50}$$

where $c_2 = (\alpha - 1)/2\Gamma(2-\alpha) \sin(\frac{\pi}{2}(\alpha - 1))$.

Since $c_1 = c_2 = (1-\alpha)/2\Gamma(2-\alpha) \cos(\frac{\pi}{2}\alpha)$, (48) and (50) yield (46). ∎

It is shown in ref. 40, using fractional calculus methodology, that if: (i) the drift function is given by $f(x) = x^{2N+1}$ (N being an integer); and, (ii) the driving Lévy process is a symmetric α-self similar noise with exponent $1 \leqslant \alpha < 2$; then

$$p(x) \approx c \cdot \frac{1}{|x|^{\alpha+2N+1}} \qquad (|x| \to \infty). \tag{51}$$

Proposition 6 confirms (51) "coming the other way around" via the reverse engineering route.

Subordinate Case

Proposition 7. Assume a subordinate system driven by an α-self similar noise with amplitude a, i.e., $\Psi(\omega) = a\omega^{\alpha}$, where $a > 0$ and $0 < \alpha < 1$. Then, the asymptotics of the reverse-engineered drift function yielding stationary pdf p are given by

$$f(x) \approx \frac{1}{\Gamma(1-\alpha)} \cdot \frac{a}{x^{\alpha} p(x)} \qquad (x \to \infty). \tag{52}$$

Proof. By (45) we have

$$\tilde{g}(\omega) = \frac{\Phi(\omega)}{\omega} \cdot \tilde{p}(\omega) \approx a\omega^{-(1-\alpha)} \qquad (\omega \to +0). \tag{53}$$

Hence, by Karamata's Tauberian theorem (see ref. 50, Thm. 1.7.1);

$$G(x) \approx \frac{a}{\Gamma(2-\alpha)} \cdot x^{1-\alpha} \qquad (x \to \infty), \tag{54}$$

where G is the primitive of g ($G' = g$). This implies that

$$g(x) \approx \frac{a}{\Gamma(1-\alpha)\, x^{\alpha}} \qquad (x \to \infty), \tag{55}$$

which, in turn, implies (52). ∎

4.4. Compound Poisson Noise with Heavy-Tailed Jumps

Analogous to the previous subsection, the use of Tauberian theorems leads to explicit asymptotic formulae for the reverse engineered drift functions also when the Lévy driver is a compound Poisson process with heavy-tailed jumps:

Symmetric Case

Proposition 8. Assume a symmetric system driven by a compound Poisson noise with rate r and heavy tailed jumps $\mathbf{P}(J > x) \approx ax^{-\alpha}$ $(x \to \infty)$, where $a > 0$ and $0 < \alpha < 2$, $\alpha \neq 1$. Then, the asymptotics of the reverse-engineered drift function yielding stationary pdf p are given by

$$f(x) \approx \frac{ra}{|x|^{\alpha}\, p(x)} \qquad (|x| \to \infty). \tag{56}$$

Proof. Since $\mathbf{P}(J > x) \approx ax^{-\alpha}$ $(x \to \infty)$, Pitman's Tauberian theorem for random variables (see ref. 50, Thm. 8.1.10) implies that

$$1 - \mathbf{E}[\exp\{i\omega J\}] \approx a \frac{\pi}{\Gamma(\alpha)\sin(\frac{\pi}{2}\alpha)} \cdot \omega^{\alpha} \qquad (\omega \to +0). \tag{57}$$

Hence, since $\Psi(\omega) = r(1 - \mathbf{E}[\exp\{i\omega J\}])$ (recall (11)), plugging (57) into (42) yields

$$\int_0^{\infty} g(x)\sin(\omega x)\, dx \approx ra \frac{\pi}{2\Gamma(\alpha)\sin(\frac{\pi}{2}\alpha)} \cdot \omega^{\alpha-1} \qquad (\omega \to +0). \tag{58}$$

Now;

(i) If $0 < \alpha < 1$ then, by Pitman's Tauberian (see ref. 50, Thm. 4.10.3), (58) is equivalent to

$$g(x) \approx \frac{ra}{x^{\alpha}} \qquad (x \to \infty). \tag{59}$$

(ii) If $1 < \alpha < 2$ then differentiating (58) gives

$$\int_0^{\infty} xg(x)\cos(\omega x)\, dx \approx ra \frac{\pi}{2\Gamma(\alpha-1)\cos(\frac{\pi}{2}(\alpha-1))} \cdot \omega^{(\alpha-1)-1} \qquad (\omega \to +0), \tag{60}$$

and, again, by Pitman's Tauberian, (60) is equivalent to

$$g(x) \approx \frac{ra}{x^{\alpha}} \qquad (x \to \infty). \tag{61}$$

Finally, (59) and (61) yield (56). ∎

Subordinate Case

Proposition 9. Assume a subordinate system driven by a compound Poisson noise with rate r and heavy tailed jumps $\mathbf{P}(J > x) \approx ax^{-\alpha}$ $(x \to \infty)$, where $0 < \alpha < 1$. Then, the asymptotics of the reverse-engineered drift function yielding stationary pdf p are given by

$$f(x) \approx \frac{ra}{x^{\alpha} p(x)} \qquad (x \to \infty). \tag{62}$$

Proof. Since $\mathbf{P}(J > x) \approx ax^{-\alpha}$ $(x \to \infty)$, Karamata's Tauberian theorem for random variables (see ref. 50, Thm. 8.1.7) implies that

$$1 - \mathbf{E}[\exp\{-\omega J\}] \approx a\Gamma(1-\alpha) \cdot \omega^{\alpha} \qquad (\omega \to +0). \tag{63}$$

Hence, since $\Phi(\omega) = r(1 - \mathbf{E}[\exp\{-\omega J\}])$ (recall (12)), plugging (63) into (45) yields

$$\tilde{g}(\omega) \approx ra\Gamma(1-\alpha) \cdot \omega^{-(1-\alpha)} \qquad (\omega \to +0). \tag{64}$$

Now; by Karamata's Tauberian theorem (see ref. 50, Thm. 1.7.1), (64) implies that

$$G(x) \approx \frac{ra}{1-\alpha} \cdot x^{1-\alpha} \qquad (x \to \infty), \tag{65}$$

where G is the primitive of g $(G' = g)$. This, in turn, implies that

$$g(x) \approx \frac{ra}{x^{\alpha}} \qquad (x \to \infty), \tag{66}$$

from which (62) follows. ∎

4.5. Subordinate Systems: Exponential Equilibria

When an exponential stationary behavior is set as the "target" pdf of a subordinate system, i.e., when

$$p(x) = \mu \cdot \exp\{-\mu x\}, \tag{67}$$

($x > 0$; $\mu > 0$), then the reverse engineering formula (38) admits a simple and explicit form:

Proposition 10. Assume a subordinate system. The system's stationary distribution is exponential (with parameter μ) if and only if the drift function f is given by

$$f(x) = \int_0^x \Lambda(u) \exp\{\mu u\}\, du. \tag{68}$$

Moreover, if $\lim_{x\to\infty} \Lambda(x) \exp\{\mu x\} = \infty$ and $\lim_{x\to\infty} \lambda(x)/\Lambda(x) = l < \mu$ then the asymptotics of the reverse-engineered drift function f are given by

$$f(x) \approx \frac{1}{\mu - l} \Lambda(x) \exp\{\mu x\} \qquad (x \to \infty). \tag{69}$$

Proof. (68) follows immediately from (40). L'Hospital's rule gives (69). ∎

We mention a few examples of jump measures satisfying the asymptotic conditions posed in the second part of Proposition 10:

1. If $\Lambda(x) \approx a x^{-\alpha}$ ($x \to \infty$; $a, \alpha > 0$) then $l = 0$.
2. If $\Lambda(x) \approx \exp\{-\eta x\}\, x^{\nu}$ ($x \to \infty$; $\eta > 0$) then $l = \eta$.
3. If $\Lambda(x) \approx \exp\{-\eta x^{\gamma}\}\, x^{\nu}$ ($x \to \infty$; $\eta > 0$, $0 < \gamma < 1$) then $l = 0$.

5. EXAMPLES

5.1. Symmetric Systems Driven by Cauchy Noise

When the driving noise is a Cauchy process with amplitude a, i.e., $\Psi(\omega) = a\,|\omega|$, then $\frac{\Psi(\omega)}{\omega} = a \cdot sign(\omega)$. This implies that in symmetric systems results regarding reverse engineering and computation of the stationary pdf obtain a special form. Indeed;

Reverse Engineering

The reverse engineering formula (44) becomes:

$$f(x) = a \cdot \frac{\int_0^\infty \hat{p}(\omega) \sin(\omega x)\, d\omega}{\int_0^\infty \hat{p}(\omega) \cos(\omega x)\, d\omega}. \tag{70}$$

Polynomial Drift

The system's drift function is given by the polynomial $f(x) = \sum_{n=1}^N c_n \cdot x^{2n-1}$ if and only if the system's stationary pdf, $\hat{p}$, satisfies the linear ODE with constant coefficients:

$$\sum_{n=1}^N c_n (-1)^n \cdot \frac{\partial^{2n-1}}{\partial \omega^{2n-1}} \hat{p}(\omega) = a \cdot \hat{p}(\omega). \tag{71}$$

5.2. Reverse Engineering: Symmetric Systems Driven by Self Similar Noise

Assume a symmetric system driven by an α-self similar noise with amplitude a, i.e., $\Psi(\omega) = a\,|\omega|^\alpha$, $0 < \alpha < 2$. In this subsection we wish to address the following reverse engineering question:

How should we reverse engineer the system so that to obtain a β-self similar stationary Lévy pdf with amplitude b, i.e., $\hat{p}(\omega) = \exp\{-b\,|\omega|^\beta\}$, $0 < \beta < 2$?

Using (44) and some basic calculus, we obtain

$$f(x) = \frac{a}{\beta b}\, x \cdot \frac{\int_0^\infty \exp\{-b\omega^\beta\}\, \omega^{\alpha-1} \sin(\omega x)\, d\omega}{\int_0^\infty \exp\{-b\omega^\beta\}\, \omega^{\beta-1} \sin(\omega x)\, d\omega}, \tag{72}$$

or, equivalently;

$$f(x) = \frac{a}{\beta b}\, x^{1+\beta-\alpha} \cdot \frac{\int_0^\infty \exp\{-b(\frac{u}{x})^\beta\}\, u^{\alpha-1} \sin(u)\, du}{\int_0^\infty \exp\{-b(\frac{u}{x})^\beta\}\, u^{\beta-1} \sin(u)\, du}. \tag{73}$$

In particular, when $\alpha = \beta$ then the drift is linear

$$f(x) = \frac{a}{\beta b}\, x,$$

and we obtain linear dynamics of the Orenstein–Uhlenbeck type.[(1)]

Cauchy Targets

In the special case where the "target" pdf is Cauchy (that is; $\beta = 1$) then the following tractable and closed form formula is obtained:

$$f(x) = ab^{1-\alpha}\Gamma(\alpha)\cdot\left(\sqrt{1+\left(\frac{x}{b}\right)^2}\right)^{2-\alpha}\cdot\sin\left(\alpha\,\mathrm{arc}\tan\left(\frac{x}{b}\right)\right). \tag{74}$$

The asymptotics of (74) are given by

$$f(x) \approx \begin{cases} \frac{a}{b^\alpha}\alpha\Gamma(\alpha)\cdot x & x\to 0 \\ \frac{a}{b}\Gamma(\alpha)\sin\left(\frac{\pi}{2}\alpha\right)\cdot x^{2-\alpha} & x\to\infty. \end{cases} \tag{75}$$

Hence, as $x\to\infty$: (i) if $\alpha<1$ then $f(x)\approx c\cdot x^p$ with $p>1$ (asymptotically convex drift); (ii) if $\alpha>1$ then $f(x)\approx c\cdot x^p$ with $p<1$ (asymptotically concave drift); (iii) if $\beta=\alpha$ then $f(x)=\frac{a}{\beta b}x$ (linear drift).

5.3. Reverse Engineering: Subordinate Systems Driven by Self Similar Noise

In this subsection we investigate subordinate systems driven by an α-self similar noise. Namely, we assume that the jump measure is given by $\lambda(dx)=\alpha\cdot x^{-(1+\alpha)}\,dx$, $x>0$, and hence $\Lambda(x)=x^{-\alpha}$, $x>0$. Using

$$(\Lambda * p)(x) = \int_0^x \frac{p(x-u)}{u^\alpha}\,du = \int_0^x \frac{p(u)}{(x-u)^\alpha}\,du,$$

together with some basic integration techniques, we obtain the following reverse engineering examples:

Pareto Equilibria

The system's stationary distribution is *Pareto*(β) ($\beta>0$) i.e.

$$p(x) = \frac{\beta}{(1+x)^{1+\beta}},$$

if and only if the drift function f is given by

$$f(x) = (1+x)^{1-\alpha}\cdot\int_0^x \frac{(1+u)^{\alpha+\beta-1}}{u^\alpha}\,du. \tag{76}$$

The asymptotic behavior of the drift function (76) is given by

$$f(x) \approx \begin{cases} \frac{1}{1-\alpha} x^{1-\alpha} & x \to 0 \\ \frac{1}{\beta} x^{1+\beta-\alpha} & x \to \infty. \end{cases}$$

Hence, as $x \to \infty$: (i) if $\beta < \alpha$ then $f(x) \approx \frac{1}{\beta} x^p$ with $p < 1$ (asymptotically concave drift); (ii) if $\beta > \alpha$ then $f(x) \approx \frac{1}{\beta} x^p$ with $p > 1$ (asymptotically convex drift); (iii) if $\beta = \alpha$ then $f(x) \approx \frac{1}{\beta} x$ (asymptotically linear drift).

In the special case $\beta = N + 1 - \alpha$, N being a non-negative integer, the drift function (76) becomes

$$f(x) = (x(1+x))^{1-\alpha} \cdot \sum_{n=0}^{N} \binom{N}{n} \frac{x^n}{n+1-\alpha}. \tag{77}$$

Exponential Equilibria

The system's stationary distribution is Exp(μ) ($\mu > 0$), i.e.

$$p(x) = \mu \cdot \exp\{-\mu x\},$$

if and only if the drift function f is given by

$$f(x) = \int_0^x \frac{\exp\{\mu u\}}{u^\alpha} du. \tag{78}$$

The asymptotic behavior of the drift function is given by

$$f(x) \approx \begin{cases} \frac{1}{1-\alpha} x^{1-\alpha} & x \to 0 \\ \frac{1}{\mu} x^{-\alpha} \exp\{\mu x\} & x \to \infty. \end{cases}$$

Gamma Equilibria

The system's stationary distribution is *Gamma*(μ, ν) ($\mu, \nu > 0$), i.e.

$$p(x) = \frac{\mu^\nu}{\Gamma(\nu)} \exp\{-\mu x\} \cdot x^{\nu-1},$$

if and only if the drift function f is given by

$$f(x) = x^{1-\alpha} \cdot \Gamma(\nu) \sum_{n=0}^{\infty} \frac{\Gamma(1-\alpha+n)}{\Gamma(\nu+1-\alpha+n)} \cdot \frac{(\mu x)^n}{n!}. \tag{79}$$

In particular, when $\nu = 1$ (that is, when the system's stationary distribution is $\mathrm{Exp}(\mu)$) then

$$f(x) = x^{1-\alpha} \cdot \sum_{n=0}^{\infty} \frac{1}{1-\alpha+n} \cdot \frac{(\mu x)^n}{n!}, \tag{80}$$

which is the power expansion of (78).

Generalized Lévy-1/2 Equilibria

The system's stationary distribution is of the type (μ, $\beta > 0$):

$$p(x) = \frac{\mu^\beta}{\Gamma(\beta)} \cdot \frac{\exp\{-\frac{\mu}{x}\}}{x^{1+\beta}},$$

if and only if the drift function f is given by

$$f(x) = x^{1-\alpha} \cdot \int_0^\infty \exp\left\{-\left(\frac{\mu}{x}\right) u\right\} \frac{(1+u)^{\alpha+\beta-1}}{u^\alpha} \, du. \tag{81}$$

In the special case $\beta = N + 1 - \alpha$, N being a non-negative integer, the drift function (81) becomes

$$f(x) = \frac{x^{2(1-\alpha)}}{\mu^{1-\alpha}} \cdot \sum_{n=0}^{N} \binom{N}{n} \Gamma(n+1-\alpha) \left(\frac{x}{\mu}\right)^n. \tag{82}$$

In particular, when $\alpha = \frac{1}{2} = \beta$ the drift function is linear

$$f(x) = \frac{\sqrt{\pi}}{\sqrt{\mu}} \cdot x.$$

and we obtain linear dynamics of the Orenstein–Uhlenbeck type.[1]

Generalized Weibull Equilibria

The system's stationary distribution is of the type (γ, μ, $\nu > 0$):

$$p(x) = \frac{\gamma \mu^\nu}{\Gamma(\frac{\nu}{\gamma})} \exp\{-(\mu x)^\gamma\} \cdot x^{\nu-1},$$

if and only if the drift function f is given by

$$f(x) = \Gamma(1-\alpha) \cdot x^{1-\alpha} \cdot g((\mu x)^\gamma), \tag{83}$$

where

$$g(y) = \exp(y) \cdot \sum_{n=0}^{\infty} \frac{\Gamma(1+\nu+\gamma n)}{\Gamma((1+\nu+\gamma n)+(1-\alpha))} \cdot \frac{(-y)^n}{n!}. \tag{84}$$

5.4. Reverse Engineering: Subordinate Systems Driven by a Compound Poisson Noise

Assume a subordinate system driven by a Lévy subordinate process with jump measure

$$\lambda(dx) = \exp\{-\mu x\} \cdot x^{N-1}\, dx,$$

where μ is a positive constant and N is an integer.

Since $\int_0^\infty \lambda(dx) = (N-1)!/\mu^N$, the driver is a compound Poisson with rate $r = (N-1)!/\mu^N$ and Gamma-distributed jumps: $J \sim Gamma(\mu, N)$ (equivalently, the jump size J equals, in distribution, a sum of N independent exponential random variables, each distributed Exp(μ)).

Using the reverse engineering formula together with some basic integration techniques, we obtain the following examples:

Exponential Equilibria

The system's stationary distribution is Exp(μ) if and only if the drift function f is given by the polynomial

$$f(x) = \frac{(N-1)!}{\mu^{N+1}} \cdot \sum_{n=1}^{N} \frac{(\mu x)^n}{n!}. \tag{85}$$

Gamma Equilibria

The system's stationary distribution is $Gamma(\mu, \nu)$ if and only if the drift function f is given by the polynomial

$$f(x) = x \cdot \frac{(N-1)!}{\mu^N} \cdot \sum_{n=0}^{N-1} \frac{(\mu x)^n}{\nu(\nu+1)(\nu+2)\cdots(\nu+n)}. \tag{86}$$

In particular, if $N=1$ (that is, if the jump size is exponential: $J \sim$ Exp(μ)) then the polynomial drift (86) reduces to the linear drift

$$f(x) = \frac{1}{\nu\mu} \cdot x.$$

and we obtain linear dynamics of the Orenstein–Uhlenbeck type.[1]

6. APPENDIX

Lemma 11. The tail function and the Fourier and Laplace characteristic of a pure-jump Lévy process L are related to each other via $\Psi(\omega)=-i\omega\hat{\Lambda}(\omega)$, and, in the subordinate case, via $\Phi(\omega)=\omega\tilde{\Lambda}(\omega)$.

Proof. General case:

$$
\begin{aligned}
-\frac{\Psi(\omega)}{i\omega} &= -\int_{-\infty}^{\infty}\frac{1-\exp\{i\omega y\}}{i\omega}\lambda(dy)\\
&= -\int_{-\infty}^{0}\frac{1-\exp\{i\omega y\}}{i\omega}\lambda(dy)+\int_{0}^{\infty}\frac{\exp\{i\omega y\}-1}{i\omega}\lambda(dy)\\
&= -\int_{-\infty}^{0}\left(\int_{y}^{0}\exp\{i\omega x\}\,dx\right)\lambda(dy)+\int_{0}^{\infty}\left(\int_{0}^{y}\exp\{i\omega x\}\,dx\right)\lambda(dy)\\
&= \int_{-\infty}^{0}\exp\{i\omega x\}\left(-\int_{-\infty}^{x}\lambda(dy)\right)dx+\int_{0}^{\infty}\exp\{i\omega x\}\left(\int_{x}^{\infty}\lambda(dy)\right)dx\\
&= \int_{-\infty}^{\infty}\exp\{i\omega x\}\,\Lambda(x)\,dx\\
&= \hat{\Lambda}(\omega).
\end{aligned}
$$

Subordinate case:

$$
\begin{aligned}
\frac{\Psi(\omega)}{\omega} &= \int_{0}^{\infty}\frac{1-\exp\{-\omega y\}}{\omega}\lambda(dy)\\
&= \int_{0}^{\infty}\left(\int_{0}^{y}\exp\{-\omega x\}\,dx\right)\lambda(dy)\\
&= \int_{0}^{\infty}\exp\{-\omega x\}\left(\int_{x}^{\infty}\lambda(dy)\right)dx\\
&= \int_{0}^{\infty}\exp\{-\omega x\}\,\Lambda(x)\,dx\\
&= \tilde{\Lambda}(\omega). \quad \blacksquare
\end{aligned}
$$

6.1. Proof of the "Key Theorems" 1 and 2

Proof. Throughout the proof $\langle \varphi_1, \varphi_2 \rangle$ will denote the inner product of the functions φ_1, φ_2 (defined on the real line);

$$\langle \varphi_1, \varphi_2 \rangle = \int_{-\infty}^{\infty} \varphi_1(u)\, \varphi_2(u)\, du.$$

We fix an exponential function $\varphi(x) = \exp\{i\omega x\}$, a pdf p, and divide the proof into four steps.

Step 1. The differential part of **A**;
Using the Fourier inversion formula for the drift function f we have

$$\begin{aligned}(\mathbf{D}\varphi)(x) &= -f(x)\, \varphi'(x) \\ &= -\left(\frac{1}{2\pi}\int_{-\infty}^{\infty} \exp\{-iux\}\, \hat{f}(u)\, du\right)\cdot (i\omega \exp\{i\omega x\}) \\ &= -\frac{i\omega}{2\pi}\int_{-\infty}^{\infty} \exp\{i(\omega - u)\, x\}\, \hat{f}(u)\, du, \end{aligned} \tag{87}$$

and hence, using basic Fourier analysis, we obtain

$$\begin{aligned}\langle \mathbf{D}\varphi, p \rangle &= -\frac{i\omega}{2\pi}\int_{-\infty}^{\infty} \hat{p}(\omega - u)\, \hat{f}(u)\, du \\ &= -\frac{i\omega}{2\pi}(\hat{p} * \hat{f})(\omega) \\ &= -i\omega(\widehat{f \cdot p})(\omega). \end{aligned} \tag{88}$$

Step 2. The integral part of **A**;
Using (9) we have

$$\begin{aligned}(\mathbf{J}\varphi)(x) &= \int_{-\infty}^{\infty} (\varphi(x+u) - \varphi(x))\, \lambda(du) \\ &= \int_{-\infty}^{\infty} (\exp\{i\omega(x+y)\} - \exp\{i\omega x\})\, \lambda(du) \\ &= -\int_{-\infty}^{\infty} (1 - \exp\{i\omega y\})\, \lambda(du) \cdot \exp\{i\omega x\} \\ &= -\Psi(\omega) \cdot \exp\{i\omega x\}, \end{aligned} \tag{89}$$

and hence, using (14) together with basic Fourier analysis, we obtain

$$\begin{aligned}\langle \mathbf{J}\varphi, p\rangle &= -\Psi(\omega)\cdot \hat{p}(\omega)\\ &= i\omega\hat{\Lambda}(\omega)\cdot\hat{p}(\omega)\\ &= i\omega(\widehat{\Lambda * p})(\omega).\end{aligned} \tag{90}$$

Step 3. The infinitesimal generator $\mathbf{A}$;
Combining (88) and (90) together yields

$$\begin{aligned}\langle \mathbf{A}\varphi, p\rangle &= \langle \mathbf{D}\varphi, p\rangle + \langle \mathbf{J}\varphi, p\rangle\\ &= -i\omega(\widehat{f\cdot p - \Lambda * p})(\omega),\end{aligned} \tag{91}$$

and hence

$$\begin{aligned}\langle \mathbf{A}\varphi, p\rangle &= \left(\widehat{\frac{\partial}{\partial x}(f\cdot p - \Lambda * p)}\right)(\omega)\\ &= \left\langle \varphi, \frac{\partial}{\partial x}(f\cdot p - \Lambda * p)\right\rangle\\ &= \langle \varphi, \mathbf{A}^* p\rangle.\end{aligned} \tag{92}$$

Step 4. Proof of Theorems 1 and 2;
Since the choice of the exponential φ was arbitrary, (92) holds for all exponential functions, and hence for all trigonometric polynomials. This, in turn, implies that the adjoint operator $\mathbf{A}^*$ is given by

$$\mathbf{A}^* p = \frac{\partial}{\partial x}(f\cdot p - \Lambda * p),$$

which proves Theorem 1.

The pdf p is a stationary pdf for the system (2) if and only if $\langle \mathbf{A}\phi, p\rangle = 0$ for all trigonometric polynomials ϕ. Since the family of exponentials spans the linear space of trigonometric polynomials, it follows from (91) that p is a stationary if and only if $(\widehat{f\cdot p - \Lambda * p})(\omega) = 0$, $\forall\omega$. Hence, p is a stationary if and only if

$$f\cdot p = \Lambda * p,$$

which proves Theorem 2. ∎

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